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An almost sure central limit theorem for self-normalized partial sums ☆

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ABSTRACT

Let $\{X_i, i \geq 1\}$ be a sequence of i.i.d. random variables which is in the domain of attraction of the normal law with mean zero and possibly infinite variance. Denote $S_n = \sum_{i=1}^n X_i$, $V_n^2 = \sum_{i=1}^n X_i^2$. Then an almost sure central limit theorem for self-normalized partial sums S_n/V_n is studied under a mild condition in this paper.

1. Introduction and main results

Recently, there has been a lot of work on the almost sure limit theorems for i.i.d. random variables. Let $\{X_i, i \geq 1\}$ be a sequence of i.i.d. random variables with mean zero, and denote $S_n = \sum_{i=1}^n X_i$. If the second moment of X_i exists then the central limit theorem holds, i.e., if $\mathsf{E}X^2 = 1$, then we have for any $x \in R$,

$$\mathsf{P}\left(\frac{S_n}{\sqrt{n}} \leq x\right) \longrightarrow \Phi(x) \tag{1.1}$$

as $n \to \infty$, where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \mathrm{e}^{-t^2/2} \mathrm{d}t. \tag{1.2}$$

The almost sure version of this limit result has been studied by many authors. Among of them, Lacey and Philipp [1] obtained that

$$\frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} I\left\{\frac{S_k}{\sqrt{k}} \leq x\right\} \longrightarrow \Phi(x) \quad \text{a.s.} \tag{1.3}$$

as $n \to \infty$ for any $x \in R$, where $I(\cdot)$ denotes the indicator function. Laterly, Fahrner and Stadtmüller [2] and Cheng et al. [3] extended the almost sure central limit theorem for partial sums to the case of maxima of i.i.d. random variables. They proved

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that for any $x \in R$,

$$\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}I\left\{\frac{M_k-b_k}{a_k}\le x\right\}\longrightarrow G(x)\quad \text{a.s.} \tag{1.4}$$

as $n\to\infty$, where $M_k=\max_{1\le i\le k}X_i$, $a_k>0$ and $b_k\in R$ satisfy

$$\mathsf{P}\left(\frac{M_k-b_k}{a_k}\le x\right)\longrightarrow G(x) \tag{1.5}$$

with $G(x)$ being one of the extreme value distributions.

It is worth mentioning that some people were interested in the almost sure central limit theorem for dependent random variables, for instance, Peligrad and Shao [4] proved results for this type for associated random variables and α-mixing random variables under some regularity conditions, and Matuła [5] considered the result for positively dependent random variables under a mild condition about the covariance of the sequence.

Compared with the classical limit theorems, the so-called self-normalized limit theorems have received more and more attention in the past decades, since the latter require little or no moment assumption if the normalizing constants in the classical limit theorems are replaced by some appropriate sequences of random variables. Among some of these beautiful results via self-normalization, Giné et al. [6] proved that

$$S_n/V_n\longrightarrow N(0,1)\text{ in distribution} \tag{1.6}$$

as $n\to\infty$ is equivalent to $\{X_i, i\ge 1\}$ is a sequence of i.i.d. random variables and in the domain of attraction of the normal law with mean zero. Here and in what follows we denote $V_n^2=\sum_{i=1}^n X_i^2$. In addition, we refer to [7] for the self-normalized law of the iterated logarithm, Shao [8] for self-normalized large deviations without moment conditions, Csörgő et al. [9] for Donsker's theorem, Jing et al. [10] for Cramér type large deviations, Jing et al. [11] for saddlepoint approximation with no moment conditions, and Zhou and Jing [12] for tail probability approximation. For a survey on recent developments in this area, the reader is referred to [13] or [14] for details.

Naturally, we want to ask whether the almost sure central limit theorem holds for self-normalized partial sums S_n/V_n. An affirmative answer will be given under a mild condition in this paper. Throughout this paper, we denote C the positive constant whose value can differ from line to line, and C_1 stands for a universal constant. Now, we give our main result.

Theorem 1.1. *Let $\{X_i, i\ge 1\}$ be a sequence of i.i.d. random variables and in the domain of attraction of the normal law with mean zero. If there exists a small $0<\varepsilon_0<1$ such that for large n, $n\mathsf{P}(|X_1|>\eta_n)\le C_1(\log n)^{-\varepsilon_0}$, where η_n is defined in (2.1) and (2.2). Then*

$$\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}I\left\{\frac{S_k}{V_k}\le x\right\}\longrightarrow \Phi(x)\quad \text{a.s.} \tag{1.7}$$

as $n\to\infty$ for any $x\in R$.

Remark 1. It follows from the definition of η_n and Lemma 2.1 that $n\mathsf{P}(|X_1|>\eta_n)=o(1)$. Hence, it seems that $n\mathsf{P}(|X_1|>\eta_{1n})\le C_1(\log n)^{-\varepsilon_0}$ is not a too strong a condition.

Remark 2. The key tool in the proof of the theorem is an upper bound and a lower bound for the indicator function of the self-normalized partial sums, which may be useful to prove other almost sure limit theorems for self-normalized partial sums if the classical versions have been established.

Remark 3. Berkes and Dehling [15] provided a sufficient and a necessary condition for the almost sure central limit theorem for i.i.d. random variables with possible infinite variance, however, it seems that the self-normalized version considered in this paper cannot be implied easily by their result.

2. Proof

One of the ideas of proofs is based on truncated random variables. That is, let

$$l(x)=E(X_1^2I\{|X_1|\le x\}),\qquad b=\inf\{x\ge 1: l(x)>0\} \tag{2.1}$$

and

$$\eta_j=\inf\left\{s: s\ge b+1, \frac{l(s)}{s^2}\le\frac{1}{j}\right\},\quad \text{for } j=1,2,3,\ldots. \tag{2.2}$$

Then it is easy to see that $\eta_n^2 \approx nl(\eta_n)$. For each n and $1 \le i \le n$, we denote

$$\bar{X}_{ni} = X_i I\{|X_i| \le \eta_n\}, \qquad \bar{S}_n = \sum_{i=1}^n \bar{X}_{ni}, \qquad \bar{V}_n^2 = \sum_{i=1}^n \bar{X}_{ni}^2. \tag{2.3}$$

Furthermore, the following two lemmas will be useful in the proof, and the first is due to Csörgő et al. [9].

Lemma 2.1. *Let X be a random variable, and denote $l(x) = EX^2 I\{|X| \le x\}$. The following statements are equivalent:*

(a) *X is in the domain of attraction of the normal law,*
(b) *$x^2 P(|X| > x) = o(l(x))$,*
(c) *$xE(|X|I\{|X| > x\}) = o(l(x))$,*
(d) *$E(|X|^n I\{|X| \le x\}) = o(x^{n-2} l(x))$ for $n > 2$.*

Lemma 2.2. *Let $\{X_i, i \ge 1\}$ be a sequence of i.i.d. random variables which is in the domain of attraction of the normal law with mean zero, and $f(\cdot)$ be a non-negative, real function such that $\sup_{x\in R} |f(x)| \le C$ and $\sup_{x\in R} |f'(x)| \le C$. In addition, if there exist positive constants ε_1, ε_2 and ε_3 such that*

$$\begin{cases} \mathsf{Var}\left(\sum_{k=1}^n \frac{1}{k} f(\bar{S}_k - \mathsf{E}\bar{S}_k)/\sqrt{kl(\eta_k)}\right) = O((\log n)^{2-\varepsilon_1}), \\ \mathsf{Var}\left(\sum_{k=1}^n \frac{1}{k} \sum_{i=1}^k I\{|X_i| > \eta_k\}\right) = O((\log n)^{2-\varepsilon_2}), \\ \mathsf{Var}\left(\sum_{k=1}^n \frac{1}{k} f(\bar{V}_k^2/(kl(\eta_k)))\right) = O((\log n)^{2-\varepsilon_3}). \end{cases} \tag{2.4}$$

Then

$$\begin{cases} \frac{1}{\log n} \sum_{k=1}^n \frac{1}{k} f(\bar{S}_k - \mathsf{E}\bar{S}_k)/(\sqrt{kl(\eta_k)}) \longrightarrow \mathsf{E}f(N(0,1)) \quad \text{a.s.}, \\ \frac{1}{\log n} \sum_{k=1}^n \frac{1}{k} \sum_{i=1}^k I\{|X_i| > \eta_n\} = \lim_{k\to\infty} k\mathsf{P}(|X_1| > \eta_k) \quad \text{a.s.}, \\ \frac{1}{\log n} \sum_{k=1}^n \frac{1}{k} f(\bar{V}_k^2/(kl(\eta_k))) = \lim_{k\to\infty} \mathsf{E}f(\bar{V}_k^2/(kl(\eta_k))) \quad \text{a.s.} \end{cases} \tag{2.5}$$

Proof. The proof is similar to the arguments used in [4] for proving directly the almost sure central limit theorem for weakly dependent random variables, and we only show that the first part of (2.5) holds under the first condition of (2.4), since the proofs of the other parts are similar. Observe that $f(\cdot)$ is a non-negative, bounded Lipschitz function. From the cental limit theorem for i.i.d. random variables from the domain of attraction of the normal law, it follows that

$$\mathsf{E}f((\bar{S}_n - \mathsf{E}\bar{S}_n)/\sqrt{nl(\eta_n)}) \longrightarrow \mathsf{E}f(N(0,1)) \tag{2.6}$$

as $n \to \infty$ since

$$\mathsf{Var}(X_i I\{|X_i| \le \eta_n\}) = (1 + o(1))l(\eta_n). \tag{2.7}$$

This implies that

$$\frac{1}{\log n} \sum_{k=1}^n \frac{\mathsf{E}f((\bar{S}_k - \mathsf{E}\bar{S}_k)/\sqrt{kl(\eta_k)})}{k} \longrightarrow \mathsf{E}f(N(0,1)) \tag{2.8}$$

as $n \to \infty$. By taking $m_k = [e^{k^{2/\varepsilon_1}}]$, here and in what follows, $[a]$ denotes the integer part of a which is not large than a, then it is easy to see

$$\begin{cases} \limsup_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^n \frac{1}{k} f((\bar{S}_k - \mathsf{E}\bar{S}_k)/\sqrt{kl(\eta_k)}) = \limsup_{j\to\infty} \frac{1}{\log m_j} \sum_{k=1}^{m_j} \frac{1}{k} f((\bar{S}_k - \mathsf{E}\bar{S}_k)/\sqrt{kl(\eta_k)}) \quad \text{a.s.}, \\ \liminf_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^n \frac{1}{k} f((\bar{S}_k - \mathsf{E}\bar{S}_k)/\sqrt{kl(\eta_k)}) = \liminf_{j\to\infty} \frac{1}{\log m_j} \sum_{k=1}^{m_j} \frac{1}{k} f((\bar{S}_k - \mathsf{E}\bar{S}_k)/\sqrt{kl(\eta_k)}) \quad \text{a.s.}, \end{cases} \tag{2.9}$$

since $f(\cdot)$ is non-negative and $\lim_{j\to\infty} \log m_j/\log m_{j+1} = 1$. It follows from the first part of (2.4) and the Borel–Cantelli lemma, that one has

$$\lim_{j\to\infty} \frac{1}{\log m_j} \sum_{k=1}^{m_j} \frac{1}{k} \left(f((\bar{S}_k - \mathsf{E}\bar{S}_k)/\sqrt{kl(\eta_k)}) - \mathsf{E}(f((\bar{S}_k - \mathsf{E}\bar{S}_k)/\sqrt{kl(\eta_k)})) \right) = 0 \quad \text{a.s.} \tag{2.10}$$

which together with (2.8) and (2.9) yields

$$\frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} f((\bar{S}_k - \mathsf{E}\bar{S}_k)/\sqrt{kl(\eta_k)}) \longrightarrow \mathsf{E}f(N(0,1)) \quad \text{a.s.} \tag{2.11}$$

The lemma is proved. □

Proof of Theorem 1.1. For any given $0 < \varepsilon < 1$, it is clear that

$$\begin{aligned} I\left\{\frac{S_k}{V_k} \le x\right\} \le \max\Bigg(& I\left\{\frac{\bar{S}_k}{\sqrt{(1+\varepsilon)kl(\eta_k)}} \le x\right\} + I\{\bar{V}_k^2 > (1+\varepsilon)kl(\eta_k)\}, I\left\{\frac{\bar{S}_k}{\sqrt{(1-\varepsilon)kl(\eta_k)}} \le x\right\} \\ & + I\{\bar{V}_k^2 < (1-\varepsilon)kl(\eta_k)\}\Bigg) + I\left\{\bigcup_{i=1}^{k} \{|X_i| > \eta_k\}\right\} \end{aligned} \tag{2.12}$$

and

$$\begin{aligned} I\left\{\frac{S_k}{V_k} \le x\right\} \ge \min\Bigg(& I\left\{\frac{\bar{S}_k}{\sqrt{(1-\varepsilon)kl(\eta_k)}} \le x\right\} - I\{\bar{V}_k^2 < (1-\varepsilon)kl(\eta_k)\}, I\left\{\frac{\bar{S}_k}{\sqrt{(1+\varepsilon)kl(\eta_k)}} \le x\right\} \\ & - I\{\bar{V}_k^2 > (1+\varepsilon)kl(\eta_k)\}\Bigg) - I\left\{\bigcup_{i=1}^{k} \{|X_i| > \eta_k\}\right\}. \end{aligned} \tag{2.13}$$

Hence, it suffices to show

$$\begin{cases} \dfrac{1}{\log n} \displaystyle\sum_{k=1}^{n} \frac{1}{k} I\left\{\frac{\bar{S}_k}{\sqrt{(1\pm\varepsilon)kl(\eta_k)}} \le x\right\} \longrightarrow \Phi(\sqrt{1\pm\varepsilon}\cdot x) \quad \text{a.s.}, \\ \dfrac{1}{\log n} \displaystyle\sum_{k=1}^{n} \frac{1}{k} \sum_{i=1}^{k} I\{|X_i| > \eta_k\} \longrightarrow 0 \quad \text{a.s.}, \\ \dfrac{1}{\log n} \displaystyle\sum_{k=1}^{n} \frac{1}{k} I\{\bar{V}_k^2 > (1+\varepsilon)kl(\eta_k)\} \longrightarrow 0 \quad \text{a.s.}, \\ \dfrac{1}{\log n} \displaystyle\sum_{k=1}^{n} \frac{1}{k} I\{\bar{V}_k^2 < (1-\varepsilon)kl(\eta_k)\} \longrightarrow 0 \quad \text{a.s.} \end{cases} \tag{2.14}$$

for proving Theorem 1.1 by the arbitrariness of ε.

The first step: we assume the conditions of (2.4) are satisfied, then we will prove that (2.14) holds. To this end, let $0 < \delta < 1/2$ and $f(\cdot)$ be a real function, such that for any given $x \in R$,

$$I\{y \le \sqrt{1\pm\varepsilon}x - \delta\} \le f_x(y) = f(y) \le I\{y \le \sqrt{1\pm\varepsilon}x + \delta\}. \tag{2.15}$$

By $\mathsf{E}X_1 = 0$, Lemma 2.1 and recalling $\eta_n^2 \approx nl(\eta_n)$, we have

$$|\mathsf{E}\bar{S}_k| \le k\mathsf{E}|X_1| I\{|X_1| > \eta_k\} = o(\sqrt{kl(\eta_k)}), \tag{2.16}$$

which yields the first part of (2.14) which holds by Lemma 2.2, (2.15) and the arbitrariness of δ in (2.15). Consider the second part of (2.14). Clearly, it suffices to show $k\mathsf{P}(|X_1| > \eta_k) \to 0$ as $k \to \infty$. Indeed, this is true because of Lemma 2.1. Consider the third part of (2.14). It is sufficient to show

$$\mathsf{P}(\bar{V}_k^2 > (1+\varepsilon)kl(\eta_k)) \to 0 \tag{2.17}$$

as $k \to \infty$. In fact, Lemma 2.1 implies

$$\begin{aligned} \mathsf{P}(\bar{V}_k^2 > (1+\varepsilon)kl(\eta_k)) &= \mathsf{P}(\bar{V}_k^2 - \mathsf{E}\bar{V}_k^2 > \varepsilon kl(\eta_k)) \\ &\le \frac{k\mathsf{E}X_1^4 I\{|X_i| \le \eta_k\}}{\varepsilon^2 k^2 \eta_k^2} \end{aligned}$$

$$= \frac{k \cdot o(\eta_k^2 l(\eta_k))}{\varepsilon^2 k^2 l^2(\eta_k)} = o(1). \tag{2.18}$$

Similar arguments also yield

$$\mathsf{P}(\bar{V}_k^2 < (1-\varepsilon)kl(\eta_k)) = o(1). \tag{2.19}$$

These finish the proof of the third and fourth parts of (2.14).

The second step: we will verify the conditions of (2.4). First, we deal with the first part of (2.4). Using the independence of the sequence $\{X_i, i \geq 1\}$ and Hölder inequality, we have

$$\begin{aligned}
\mathsf{Var}\left(\sum_{k=1}^{n} \frac{1}{k} f(\bar{S}_k - \mathsf{E}\bar{S}_k)/\sqrt{kl(\eta_k)}\right) &\leq C\sum_{k=1}^{n}\frac{1}{k^2} + 2\sum_{j=2}^{n}\sum_{i=1}^{j-1}\frac{1}{ij}\cdot \mathsf{Cov}\left(f\left(\frac{\bar{S}_i - \mathsf{E}\bar{S}_i}{\sqrt{il(\eta_i)}}\right), f\left(\frac{\bar{S}_j - \mathsf{E}\bar{S}_j}{\sqrt{jl(\eta_j)}}\right)\right) \\
&\leq C + 2\sum_{j=2}^{n}\sum_{i=1}^{j-1}\frac{1}{ij}\cdot \mathsf{Cov}\Bigg(f\left(\frac{\bar{S}_i - \mathsf{E}\bar{S}_i}{\sqrt{il(\eta_i)}}\right), f\left(\frac{\bar{S}_j - \mathsf{E}\bar{S}_j}{\sqrt{jl(\eta_j)}}\right) \\
&\quad - f\left(\frac{(\bar{S}_j - \bar{S}_i) - \mathsf{E}(\bar{S}_j - \bar{S}_i)}{\sqrt{jl(\eta_j)}}\right)\Bigg) \\
&\leq C + 2\sum_{j=2}^{n}\sum_{i=1}^{j-1}\frac{1}{ij}\mathsf{E}\left|f\left(\frac{\bar{S}_j - \mathsf{E}\bar{S}_j}{\sqrt{jl(\eta_j)}}\right) - f\left(\frac{(\bar{S}_j - \bar{S}_i) - \mathsf{E}(\bar{S}_j - \bar{S}_i)}{\sqrt{jl(\eta_j)}}\right)\right| \\
&\leq C + C\sum_{j=2}^{n}\sum_{i=1}^{j-1}\frac{1}{ij}\frac{\mathsf{E}|\bar{S}_i - \mathsf{E}\bar{S}_i|}{\sqrt{jl(\eta_j)}} \\
&\leq C + C\sum_{j=2}^{n}\sum_{i=1}^{j-1}\frac{1}{ij}\frac{\sqrt{il(\eta_i)}}{\sqrt{jl(\eta_j)}} \\
&= O(\log n),
\end{aligned} \tag{2.20}$$

which completes the verification of the first part of (2.4). Second, we consider the second part of (2.4). If follows from the similar arguments of (2.20) and the condition $n\mathsf{P}(|X_1| > \eta_n) \leq C(\log n)^{-\varepsilon_0}$ from Theorem 1.1 that

$$\begin{aligned}
\mathsf{Var}\left(\sum_{k=1}^{n}\frac{1}{k}\sum_{i=1}^{k} I\{|X_i| > \eta_k\}\right) &\leq \sum_{k=1}^{n}\frac{1}{k^2}\cdot k\mathsf{P}(|X_1| > \eta_k) + 2\sum_{j=2}^{n}\sum_{i=1}^{j-1}\frac{1}{ij}\mathsf{Cov}\left(\sum_{k=1}^{i} I\{|X_k| > \eta_i\}, \sum_{k=1}^{j} I\{|X_k| > \eta_j\}\right) \\
&\leq C + 2\sum_{j=2}^{n}\sum_{i=1}^{j-1}\frac{1}{ij}\cdot i\mathsf{P}(|X_1| > \eta_i) \\
&\leq C + C\sum_{j=2}^{n}\sum_{i=1}^{j-1}\frac{1}{ij}(\log i)^{-\varepsilon_0} \\
&= O((\log n)^{2-\varepsilon_0}),
\end{aligned} \tag{2.21}$$

which completes the verification of the second part of (2.4). As for the verification of the third part of (2.4), one can easily apply the similar arguments of (2.20) to get

$$\mathsf{Var}\left(\sum_{k=1}^{n}\frac{1}{k} f(\bar{V}_k^2/(kl(\eta_k)))\right) = O(\log n) \tag{2.22}$$

and we omit the details here. The proof is completed. □

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